Exercise 1.1.11

Determine whether each of these series is convergent, and if so, whether it is absolutely convergent:

(a) $\frac{\ln 2}{2} - \frac{\ln 3}{3} + \frac{\ln 4}{4} - \frac{\ln 5}{5} + \frac{\ln 6}{6} - \cdots,$

(b) $\frac{1}{1} + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \cdots,$

(c) $1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} - \frac{1}{9} - \frac{1}{10} + \frac{1}{11} \cdots + \frac{1}{15} - \frac{1}{16} \cdots - \frac{1}{21} + \cdots.$

Solution

Part (a)

Rewrite the given series.

$$\frac{\ln 2}{2} - \frac{\ln 3}{3} + \frac{\ln 4}{4} - \frac{\ln 5}{5} + \frac{\ln 6}{6} - \cdots = \sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n}$$

To determine whether this alternating series converges, check the two conditions of the Leibniz criterion.

$$\text{(i)} \quad \frac{d}{dn}\left(\frac{\ln n}{n}\right) = \frac{1 - \ln n}{n^2} < 0 \quad \text{if } n > e$$

$$\text{(ii)} \quad \lim_{n\to\infty} \frac{\ln n}{n} \overset{\frac{\infty}{\infty}}{\underset{\text{H}}{=}} \lim_{n\to\infty} \frac{\frac{d}{dn}(\ln n)}{\frac{d}{dn}(n)} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1} = \lim_{n\to\infty} \frac{1}{n} = 0$$

Since the first derivative of $\ln n/n$ is negative (for sufficiently large n), $\ln n/n$ is a monotonically decreasing function. The limit of $\ln n/n$ is zero as $n \to \infty$. Therefore, the series converges. Consider the series with positive terms now.

$$\sum_{n=2}^{\infty} \frac{\ln n}{n}$$

$\ln n/n$ is monotonically decreasing, so use the integral test to prove or disprove convergence.

$$f(x) = \frac{\ln x}{x}$$

$\ln x$ and x are continuous functions, so their ratio is as well. In addition, $\ln x/x$ is positive from 2 to ∞. The conditions to use the integral test are satisfied; now evaluate the corresponding integral

$$\int_2^{\infty} \frac{\ln x}{x}\, dx$$

by using the substitution $u = \ln x$ $(du = dx/x)$.

$$\int_{\ln 2}^{\infty} u\, du$$

$$\left.\frac{u^2}{2}\right|_{\ln 2}^{\infty}$$

$$\infty$$

The series of positive terms diverges by the integral test. Therefore, the series in question,

$$\frac{\ln 2}{2} - \frac{\ln 3}{3} + \frac{\ln 4}{4} - \frac{\ln 5}{5} + \frac{\ln 6}{6} - \cdots = \sum_{n=2}^{\infty} (-1)^n \frac{\ln n}{n},$$

is not absolutely convergent but rather conditionally convergent.

Part (b)

Rewrite the given series.

$$\frac{1}{1} + \frac{1}{2} - \frac{1}{3} - \frac{1}{4} + \frac{1}{5} + \frac{1}{6} - \frac{1}{7} - \frac{1}{8} + \cdots = \left(\frac{1}{1} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\right) + \left(\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots\right)$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1} + \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+2}$$

$$= \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{2n+1} + \frac{1}{2n+2}\right)$$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{4n+3}{(2n+1)(2n+2)}$$

To determine whether this alternating series converges, check the two conditions of the Leibniz criterion.

(i) $$\frac{d}{dn}\left[\frac{4n+3}{(2n+1)(2n+2)}\right] = -\frac{2}{(2n+1)^2} - \frac{1}{2(n+1)^2} < 0$$

(ii) $$\lim_{n\to\infty} \frac{4n+3}{(2n+1)(2n+2)} = \lim_{n\to\infty} \frac{4n+3}{4n^2+6n+2} = \lim_{n\to\infty} \frac{\frac{4}{n} + \frac{3}{n^2}}{4 + \frac{6}{n} + \frac{2}{n^2}} = \frac{0+0}{4+0+0} = 0$$

Since the first derivative of $(4n+3)/[(2n+1)(2n+2)]$ is negative, $(4n+3)/[(2n+1)(2n+2)]$ is a monotonically decreasing function. The limit of $(4n+3)/[(2n+1)(2n+2)]$ is zero as $n \to \infty$. Therefore, the series converges.

Consider the series with positive terms now.

$$\sum_{n=0}^{\infty} \frac{4n+3}{(2n+1)(2n+2)}$$

$(4n+3)/[(2n+1)(2n+2)]$ is monotonically decreasing, so use the integral test to prove or disprove convergence. Consider the corresponding function of x.

$$f(x) = \frac{4x+3}{(2x+1)(2x+2)}$$

$2x+1$ and $2x+2$ are continuous functions, so their product is as well. $4x+3$ is also continuous, so the ratio of $4x+3$ to $(2x+1)(2x+2)$ is also continuous. $f(x)$ is positive from 0 to ∞. The conditions to use the integral test are satisfied; now evaluate the corresponding integral

$$\int_0^{\infty} \frac{4x+3}{(2x+1)(2x+2)}\, dx$$

by using partial fraction decomposition.

$$\int_0^{\infty} \left(\frac{1}{2x+1} + \frac{1}{2x+2} \right) dx$$

$$\ln(2x+1)\Big|_0^{\infty} + \frac{1}{2}\ln(x+1)\Big|_0^{\infty}$$

$$\infty$$

The series of positive terms diverges by the integral test. Therefore, the series in question,

$$\frac{\ln 2}{2} - \frac{\ln 3}{3} + \frac{\ln 4}{4} - \frac{\ln 5}{5} + \frac{\ln 6}{6} - \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{4n+3}{(2n+1)(2n+2)},$$

is not absolutely convergent but rather conditionally convergent.

Part (c)

Rewrite the given series.

$$1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}-\frac{1}{9}-\frac{1}{10}+\frac{1}{11}\cdots+\frac{1}{15}-\frac{1}{16}\cdots-\frac{1}{21}+\cdots$$

$$=\left(\sum_{n=1}^{1}\frac{1}{n}\right)-\left(\sum_{n=2}^{3}\frac{1}{n}\right)+\left(\sum_{n=4}^{6}\frac{1}{n}\right)-\left(\sum_{n=7}^{10}\frac{1}{n}\right)+\left(\sum_{n=11}^{15}\frac{1}{n}\right)-\cdots$$

Each term of this alternating series has a lower bound and an upper bound.

$$\frac{1}{1}\le\sum_{n=1}^{1}\frac{1}{n}\le\frac{1}{1}$$

$$\frac{1}{3}+\frac{1}{3}\le\sum_{n=2}^{3}\frac{1}{n}\le\frac{1}{2}+\frac{1}{2}$$

$$\frac{1}{6}+\frac{1}{6}+\frac{1}{6}\le\sum_{n=4}^{6}\frac{1}{n}\le\frac{1}{4}+\frac{1}{4}+\frac{1}{4}$$

$$\frac{1}{10}+\frac{1}{10}+\frac{1}{10}+\frac{1}{10}\le\sum_{n=7}^{10}\frac{1}{n}\le\frac{1}{7}+\frac{1}{7}+\frac{1}{7}+\frac{1}{7}$$

$$\frac{1}{15}+\frac{1}{15}+\frac{1}{15}+\frac{1}{15}+\frac{1}{15}\le\sum_{n=11}^{15}\frac{1}{n}\le\frac{1}{11}+\frac{1}{11}+\frac{1}{11}+\frac{1}{11}+\frac{1}{11}$$

Calling the kth term of the series s_k, generalize the formulas on the left and right sides.

$$\frac{2}{k+1}\le s_k\le\frac{2k}{k^2-k+2}$$

$$\frac{2}{k+1}\le s_k\le\frac{2}{k-1+\frac{2}{k}}<\frac{2}{k-1}$$

To determine whether this alternating series converges, check the two conditions of the Leibniz criterion.

$$\text{(i)}\quad \frac{d}{dk}\left(\frac{2}{k+1}\right)\le s_k'\le\frac{d}{dk}\left(\frac{2}{k-1}\right)\qquad\to\qquad -\frac{2}{(k+1)^2}\le s_k'\le-\frac{2}{(k-1)^2}$$

$$\text{(ii)}\quad \lim_{k\to\infty}\frac{2}{k+1}\le\lim_{k\to\infty}s_k\le\lim_{k\to\infty}\frac{2}{k-1}\qquad\to\qquad 0\le\lim_{k\to\infty}s_k\le 0\quad\Rightarrow\quad\lim_{k\to\infty}s_k=0$$

Since the first derivative of s_k is negative, s_k is a monotonically decreasing function. The limit of s_k is zero as $k\to\infty$ by the Squeeze theorem. Therefore, the series converges.

The series with positive terms,

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}\cdots+\frac{1}{15}+\frac{1}{16}\cdots+\frac{1}{21}+\cdots,$$

is the harmonic series, which diverges. Therefore, the series in question,

$$1-\frac{1}{2}-\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}-\frac{1}{7}-\frac{1}{8}-\frac{1}{9}-\frac{1}{10}+\frac{1}{11}\cdots+\frac{1}{15}-\frac{1}{16}\cdots-\frac{1}{21}+\cdots,$$

is not absolutely convergent but rather conditionally convergent.